Announcements

1) Quiz tomorrow over section 7.8

Series Solutions to Differential Equations

(Chapter 8)

Heat Equation: Boss Level

3 spatial dimensions

$U = U(x, y, z, t)$ is the temperature of a three-dimensional object at time t.

Heat Equation

$$\boxed{\frac{\partial u}{\partial t} = k\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\text{the Laplacian of } u}}$$

Boundary conditions –

specified later.

Temperature in a Cylinder

$$x^2+y^2 \leq 10$$

Convert to Cylindrical Coordinates

Cylindrical to Cartesian	Cartesian to Cylindrical
$x = r\cos\theta$	$\theta = \arctan\left(\frac{y}{x}\right)$
$y = r\sin\theta$	$r = \sqrt{x^2+y^2}$
$z = z$	$z = z$

Convert Heat Equation to cylindrical coordinates

We get

$$\boxed{\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

Assume temperature is independent of z and θ. Then

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \theta} = 0,$$ so we get

$$\boxed{\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)}$$

We suppose

$$u(r,\theta,z,t)$$

$$= u(r,t)$$

$$= \boxed{f(r)g(t)}$$

Now apply the heat equation:

$$\frac{\partial u}{\partial r} = f'(r)g(t), \quad \frac{\partial^2 u}{\partial r^2} = f''(r)g(t)$$

$$\frac{\partial u}{\partial t} = f(r)g'(t)$$

We get

$$g'(t)f(r)$$

$$= k\left(f''(r)g(t) + \frac{1}{r}f'(r)g(t)\right)$$

$$= kg(t)\left(f''(r) + \frac{1}{r}f'(r)\right)$$

Divide both sides by $kf(r)g(t)$

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(r) + \frac{1}{r}f'(r)}{f(r)}$$

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(r)+\frac{1}{r}f'(r)}{f(r)}$$

Since we have a function of r equal to a function of t, both sides of the equality must be constant:

$$\frac{1}{k}\frac{g'(t)}{g(t)} = \frac{f''(r)+\frac{1}{r}f'(r)}{f(r)}$$

$\boxed{= -\alpha}$ for some real number α

We get two ordinary differential equations:

$$\frac{1}{k} \frac{g'(t)}{g(t)} = -\alpha$$ easy

$$\frac{f''(r) + \frac{1}{r} f'(r)}{f(r)} = -\alpha$$

Rewriting,

$$f''(r) + \frac{1}{r} f'(r) = -\alpha f(r)$$

We have

$$f''(r) + \frac{1}{r} f'(r) + \alpha f(r) = 0$$

homogeneous, second-order,
nonconstant coefficients

By theory, there is a solution! How to find it?

Not Cauchy-Euler!

Wishful Thinking

Assume there are real numbers

$(a_n)_{n=0}^{\infty}$ with

$$\boxed{f(r) = \sum_{n=0}^{\infty} a_n r^n}$$

purely formal!

Treating f as a polynomial,

$$f'(r) = \sum_{n=0}^{\infty} a_n(n\, r^{n-1})$$

$$= \sum_{n=1}^{\infty} (a_n n\, r^{n-1})$$

$$f''(r) = \sum_{n=1}^{\infty} a_n n((n-1)\, r^{n-2})$$

$$= \sum_{n=2}^{\infty} (a_n n(n-1)\, r^{n-2})$$

To make the indices nice, multiply

$$f''(r) + \frac{1}{r} f'(r) + \alpha f(r) = 0$$

by r^2 (not r!) to get

$$\boxed{r^2 f''(r) + r f'(r) + \alpha r^2 f(r) = 0}$$

$$r^2 f''(r) = \sum_{n=2}^{\infty} \left(a_n n(n-1) r^n\right)$$

$$r f'(r) = \sum_{n=1}^{\infty} \left(a_n n r^n\right)$$

$$r^2 \alpha f(r) = \sum_{n=0}^{\infty} \alpha a_n r^{n+2}$$

reindex with $k = n+2$,

then $n = k-2$

$$r^2 \alpha f(r) = \sum_{k=2}^{\infty} \alpha a_{k-2} r^k$$

$$r^2 \alpha f(r) = \sum_{k=2}^{\infty} \alpha a_{k-2} r^k$$

Substituting n for k,

$$r^2 \alpha f(r) = \sum_{n=2}^{\infty} \alpha a_{n-2} r^n.$$

Then

$$r^2 f''(r) + r f'(r) + \alpha r^2 f(r)$$

$$= \left.\begin{aligned} &\sum_{n=2}^{\infty} \left(a_n (n)(n-1) r^n\right) \\ &+ \sum_{n=1}^{\infty} \left(a_n n r^n\right) \end{aligned}\right\} \text{combine}$$

$$+ \sum_{n=2}^{\infty} \alpha a_{n-2} r^n \quad \text{leave alone}$$

We get

$$a_1 r + \sum_{n=2}^{\infty} \left(\overbrace{n(n-1)+n}^{n^2} \right) a_n r^n$$

$$+ \sum_{n=2}^{\infty} \alpha a_{n-2} r^n$$

$$= a_1 r + \sum_{n=2}^{\infty} n^2 a_n r^n + \sum_{n=2}^{\infty} \alpha a_{n-2} r^n$$

$$= a_1 r + \sum_{n=2}^{\infty} \left(n^2 a_n - \alpha a_{n-2} \right) r^n$$

$$= 0$$